Operator Growth in Lattice Models and Connections to Anderson and Many-Body Localization

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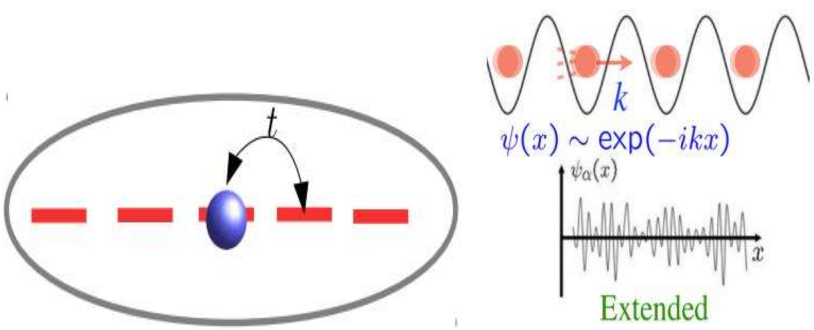
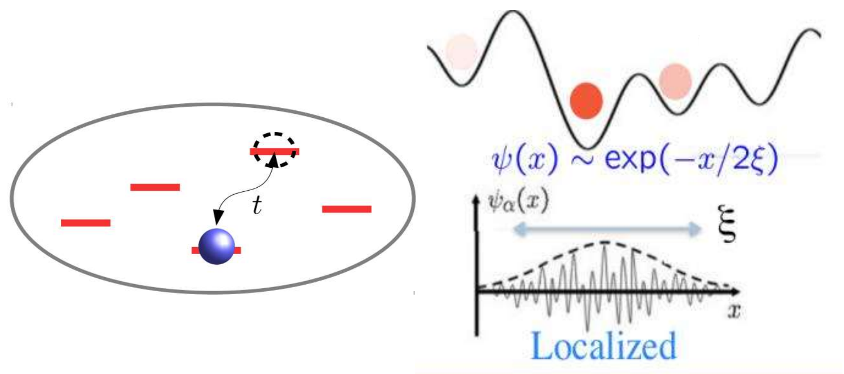
Anderson Localization

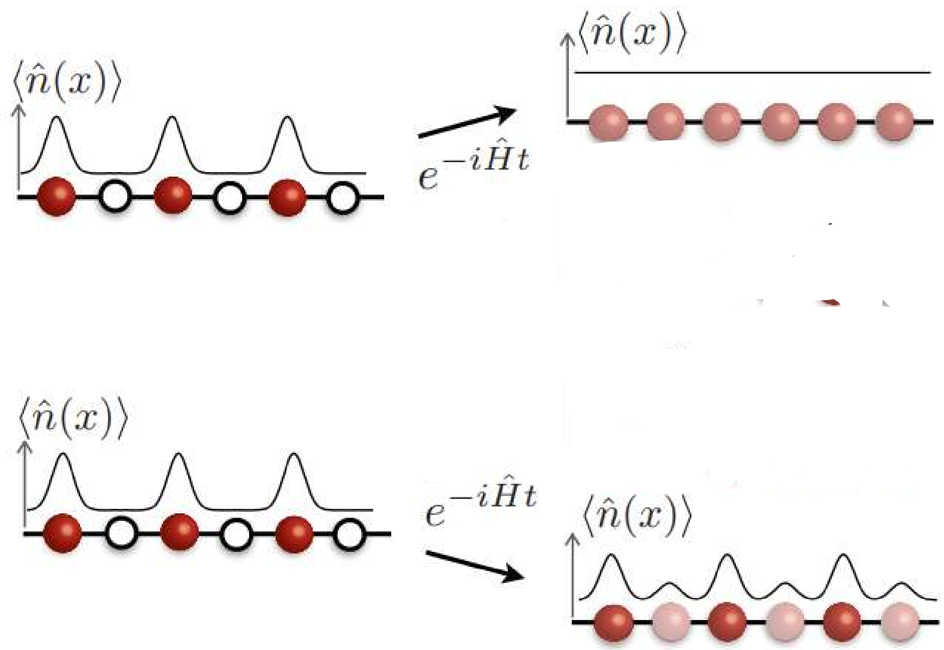
- P. Anderson (1958) considered the problem of transport in a lattice with an impurity potential (disorder)
- Expect ergodic behaviour in the regular quantum walk
- A sufficient random potential causes localized behaviour: Anderson Localization





What about interactions?

- Anderson only considered one particle
- In a **thermalizing** system:
  - Wave functions evolve as Bloch waves
  - Information in initial state erased
  - Local operators appear thermal
- In a **localized** phase:
  - Wave functions remain exponentially localized near a particular site
  - Initial information is retained (quantum memory)
  - Local operators remain local



Question: Does Anderson localization survive in the presence of interactions?

- Many authors: **yes**
  - Basko et al. (2006): perturbative arguments
  - Pal & Huse (2010) and more: exact diagonalization, numerics
  - Imbrie (2014): “proof” of MBL for 1D spin chain
- But we can’t be so sure…
  - M. Kiefer-Emmanouilidis, J. Sirker (2019-2022)
  - A. Weisse, R. Gerstner, J. Sirker (2023)

Model Considered

- Heisenberg (XXZ) spin chain

$$H = \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z + 2\, h_j \sigma_j^z)$$

  - $\sigma_j^{x,y,z}$ Pauli spin-1/2 operators at site j, Δ anisotropy parameter, h_j quasi-random magnetic field

- Jordan-Wigner transformation →

$$\frac{H}{4} = \frac{1}{2} \sum_j \{c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j\} + \sum_j \left\{\Delta \left(n_j - \frac{1}{2}\right)\left(n_{j+1} - \frac{1}{2}\right) + h_j \left(n_j - \frac{1}{2}\right)\right\}$$

- $h_j \in [-D, D]$ → localization transition at $D \approx 3.5$

Operator Growth

$$\sigma_j^z(t) = e^{itH}\sigma_j^z e^{-itH} = \sum_{k=0}^{\infty}\left[H,\sigma_j^z\right]^{(k)}\frac{(it)^k}{k!}$$

$$\left[H,\sigma_j^z\right]^{(k)} = [H,[H,\left[\dots,\left[H,\sigma_j^z\right]\right]]$$

$$||A||_1 \;=\; \sum_{i,j}|a_{i,j}|/\operatorname{tr}\mathbb{1}_{M\times M}$$

$$||A||_2 \;=\; \sqrt{\operatorname{tr}(A^\dagger A)/\operatorname{tr}\mathbb{1}_{M\times M}}$$

$\left\|\left[H,\sigma_j^z\right]^{(k)}\right\| \sim$? spatial structure?

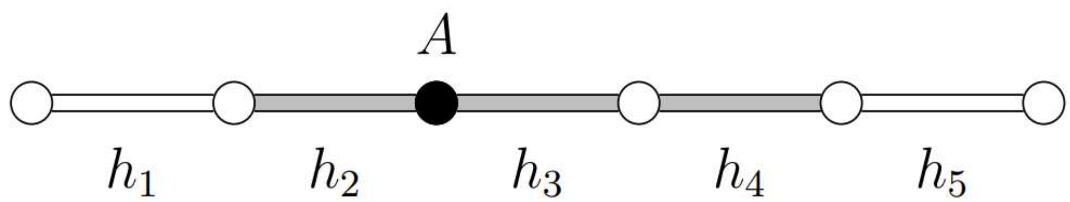
Results

1. Lattice Animal Construction
   - Following Avdoshkin & Dymarsky, 2020
2. Exact Formulae
   - Only possible in non-disordered, non-interacting case
3. Graphical Language
   - New way of visualizing and calculating growth

I. Lattice Animals

a) Setup

$$H = \sum_I h_{I,I+1} \equiv \sum_I h_I$$



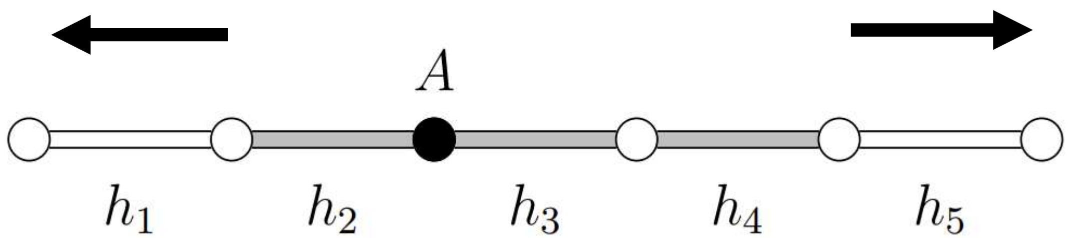
Nearest-neighbour interacting model (e.g. XXZ)

$$[H, A]^{(k)} = [\sum_I h_I, [\sum_I h_I, [\ldots, [\sum_I h_I, A]]] = \sum_{\{I_1,\ldots,I_k\}} [h_{I_k}, [h_{I_{k-1}}, \Big[\ldots, [h_{I_1}, A]\Big]\Big]]$$

$\{I_1, \ldots, I_k\}$ = set of bonds which satisfy adjacency condition = **lattice animal**

- Adjacency condition: I_1 is connected to *A*, and any subset $\{I_1, \ldots, I_l\}$, $l \leq k$, forms a connected cluster

1. Lattice Animals
b) How many lattice animals with k bonds?



- The "length" of the cluster ranges from 1 to k (no overlap)
- At a length *j*, there are 2^j ways to form the cluster (random walk), and $S(k,j)$ ways to distribute the k bonds

$$\sum_{\{I_1,\dots,I_k\}} = \sum_{j=1}^{k} 2^j S(k,j) = B_k(2)$$

- **Bell polynomial** grows asymptotically almost factorially and much faster than exponential

1. Lattice Animals

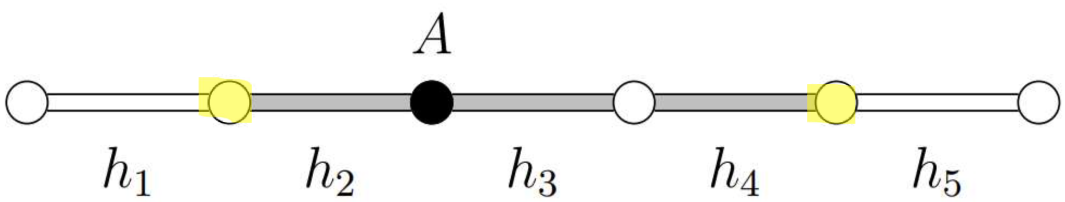
c) General bound

$$||[H,A]^{(k)}|| = ||\sum_{\{I_1,\ldots,I_k\}} [h_{I_k},[h_{I_{k-1}},\left[\ldots,\left[h_{I_1},A\right]\right]]||$$

$$\leq \sum_{\{I_1,\ldots,I_k\}} ||[h_{I_k},[h_{I_{k-1}},\left[\ldots,\left[h_{I_1},A\right]\right]]\,||$$

$$\leq 2^k \left||h|\right|^k \left||A|\right| \sum_{\{I_1,\ldots,I_k\}} \leq \left||A|\right| (2\tilde{J})^k B_k(2)$$

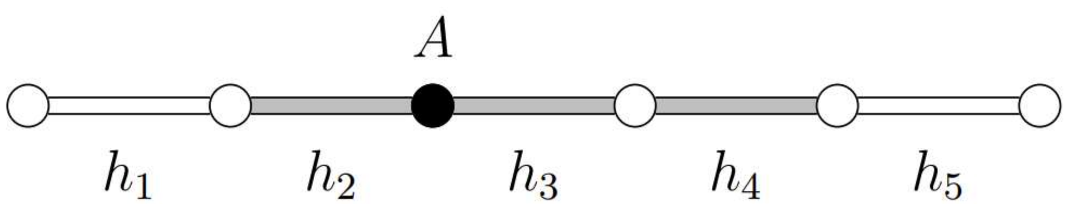
1. Lattice Animals
d) Restriction to non-interacting case



$$||[H,A]^{(k)}|| \leq \big||A|\big|(8N\tilde{J})^k$$

I. Lattice Animals
d) Localized model

- Expect exponential decay of contributions with distance from initial site



$$||[H, A]^{(k)}|| \leq 2||A||(2\lambda\tilde{J})^k.$$

2. Exact Formulae

a) Spin model

$$H = \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y)$$

$$\begin{aligned}
[H, \sigma_0^z]^{(k)} &= \sum_{l=0}^{\frac{k-1}{2}} \sum_{s=-\frac{k+1}{2}-l}^{\frac{k-1}{2}-l} a_{kls} \left\{ \sigma_s^x \bigotimes_{j=s+1}^{s+2l} \sigma_j^z \, \sigma_{s+2l+1}^y \right. \\
&\quad \left. - \; \sigma_s^y \bigotimes_{j=s+1}^{s+2l} \sigma_j^z \, \sigma_{s+2l+1}^x \right\} \\
&= -\frac{2}{i} \sum_{l=0}^{\frac{k-1}{2}} \sum_{s=-\frac{k+1}{2}-l}^{\frac{k-1}{2}-l} a_{kls} \left\{ c_s^\dagger c_{s+2l+1} - h.c. \right\}
\end{aligned}$$

$$a_{kls} = i(-1)^s (C_k)_{s+\frac{k+3}{2}+l, \frac{k+1}{2}-l}.$$

$$(C_k)_{ij} = 2^k \binom{k}{i-1} \binom{k}{j-1}$$

- Number of terms ~$(k+1)^2$

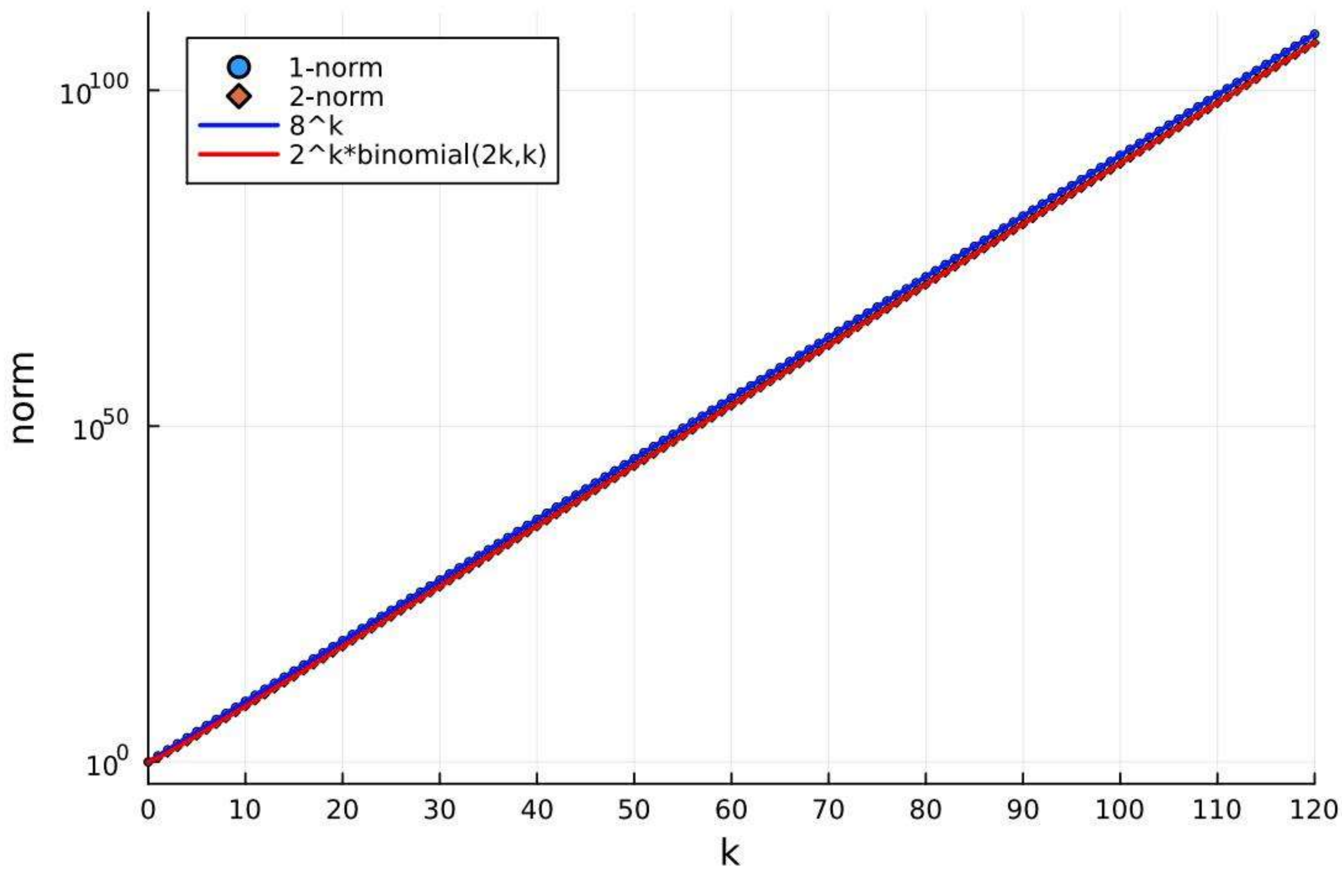
2. Exact Formulae
a) Spin model

$$(C_k)_{ij} = 2^k \binom{k}{i-1}\binom{k}{j-1}$$

- At (odd) order k, each element of C_k appears 4 times in the commutator

$$\Rightarrow \left\| [H, \sigma_0^z] \right\|_1 = 4 \sum_{i=1}^{\frac{k+1}{2}} \sum_{j=1}^{\frac{k+1}{2}} (C_k)_{ij} = 8^k$$

$$\left\| [H, \sigma_0^z] \right\|_2 = 4 \sum_{i=1}^{\frac{k+1}{2}} \sum_{j=1}^{\frac{k+1}{2}} (C_k)_{ij}^2 = 2^k \binom{2k}{k} \approx \frac{8^k}{\sqrt{\pi k}}$$


1-norm
2-norm
8^k
2^k*binomial(2k,k)
norm
10^{100}
10^{50}
10^{0}
0
10
20
30
40
50
60
70
80
90
100
110
120
k

3. Graphical Language
a) No interactions or disorder

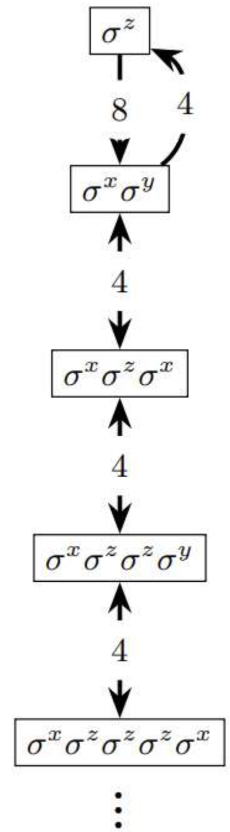
- Traversing the graph corresponds to possible commutation paths

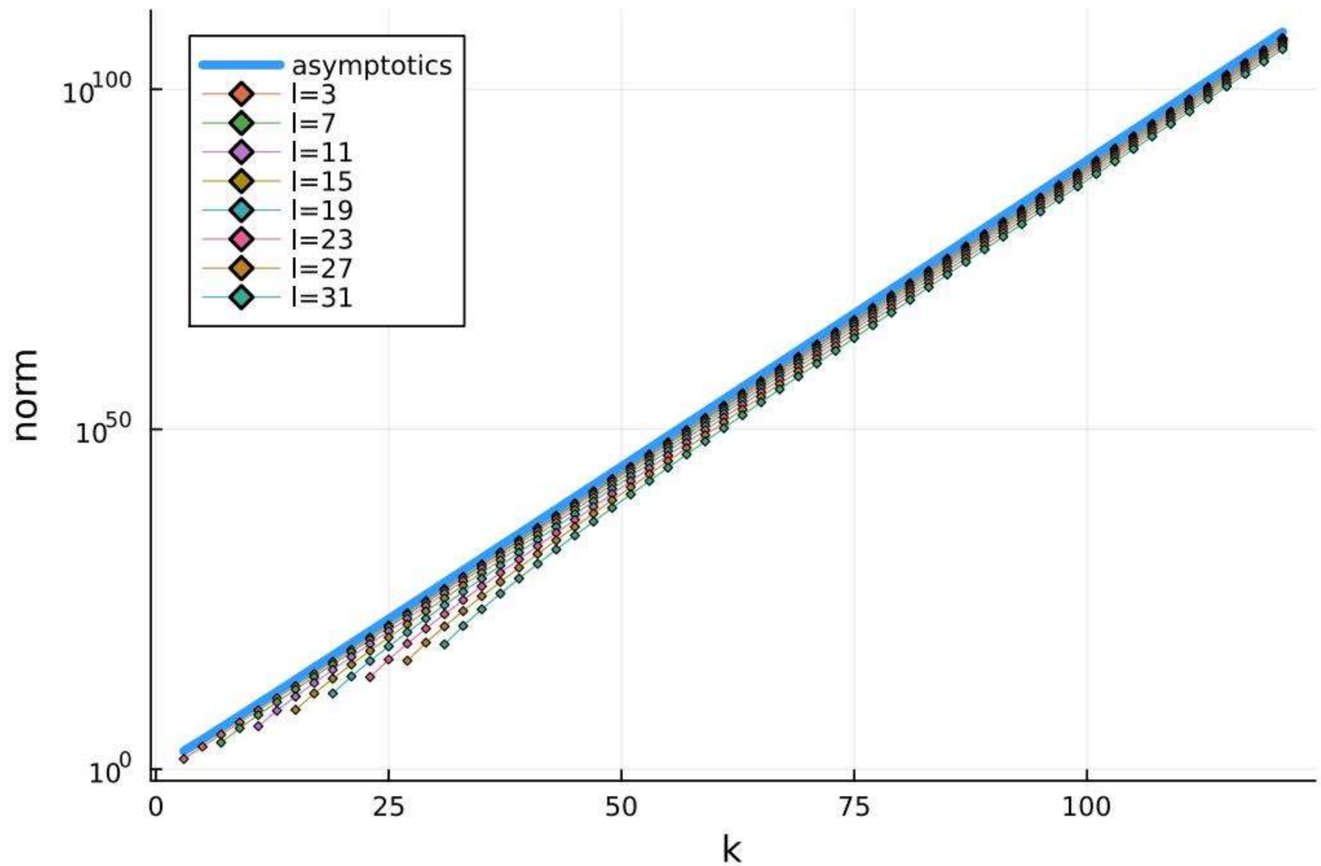
- Weighted sum over all walks starting at top node gives the 1-norm
  - k steps, each contributing $8 \Rightarrow 8^k$

- Weighted sum over all walks starting at the top node and ending at node l gives support at l sites

$$s_l(k) = 2^{2k+1}\binom{k}{\frac{k+1-l}{2}} \approx \sqrt{\frac{8}{k\pi}}8^k \ (k \gg l \gg 1)$$

- Independent of $l \Rightarrow$ thermalization!





asymptotics
l=3
l=7
l=11
l=15
l=19
l=23
l=27
l=31
norm
10^{100}
10^{50}
10^{0}
0
25
50
75
100
k

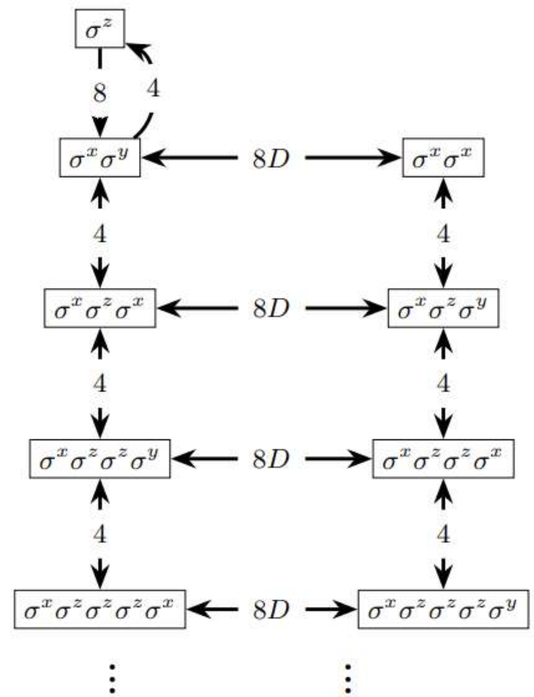
3. Graphical Language
b) Disordered, non-interacting (Anderson case)

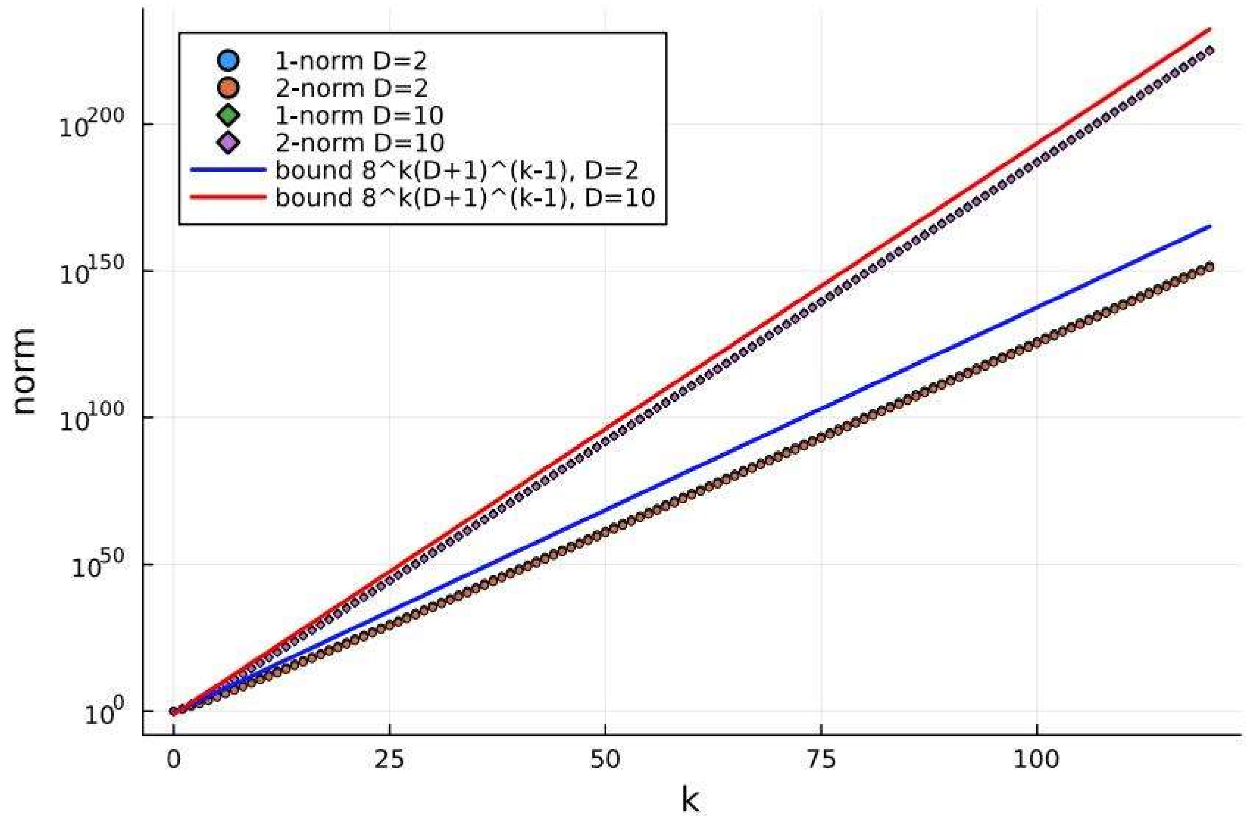
$$H = \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + 2\, h_j \sigma_j^z)$$

$$\left|\left|[H, \sigma_0^z]\right|\right|_1 \leq 8^k (D+1)^{k-1}$$

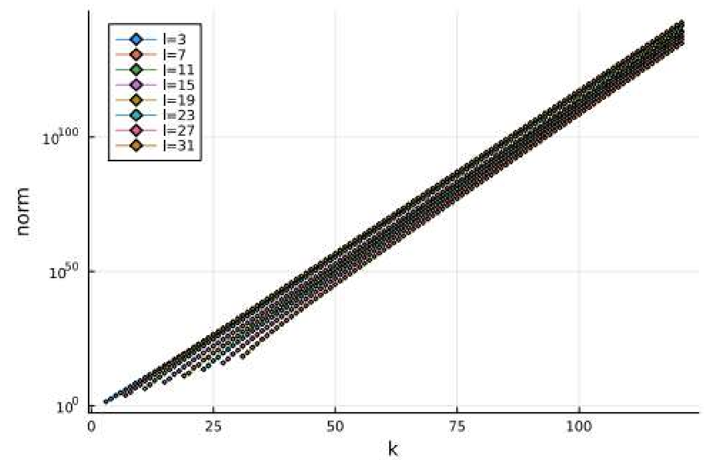
- Horizontal edges limit ability to reach more distant sites
  - Contributions of horizontal edges grow with disorder strength
  - **Localization** occurs

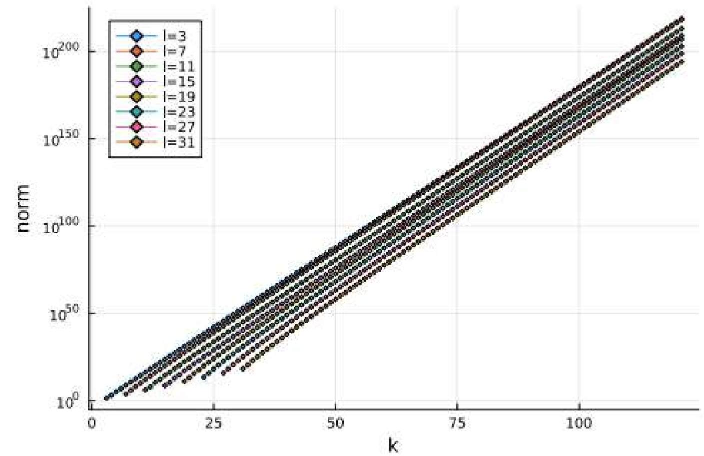
$$\frac{s_l(k)}{s_1(k)} \leq \binom{k-1}{l-2} (2D)^{3-l} \approx \frac{1}{\sqrt{2\pi l}} \left(\frac{ke}{l}\right)^l e^{-\frac{l^2}{2k}} D^{3-l}$$



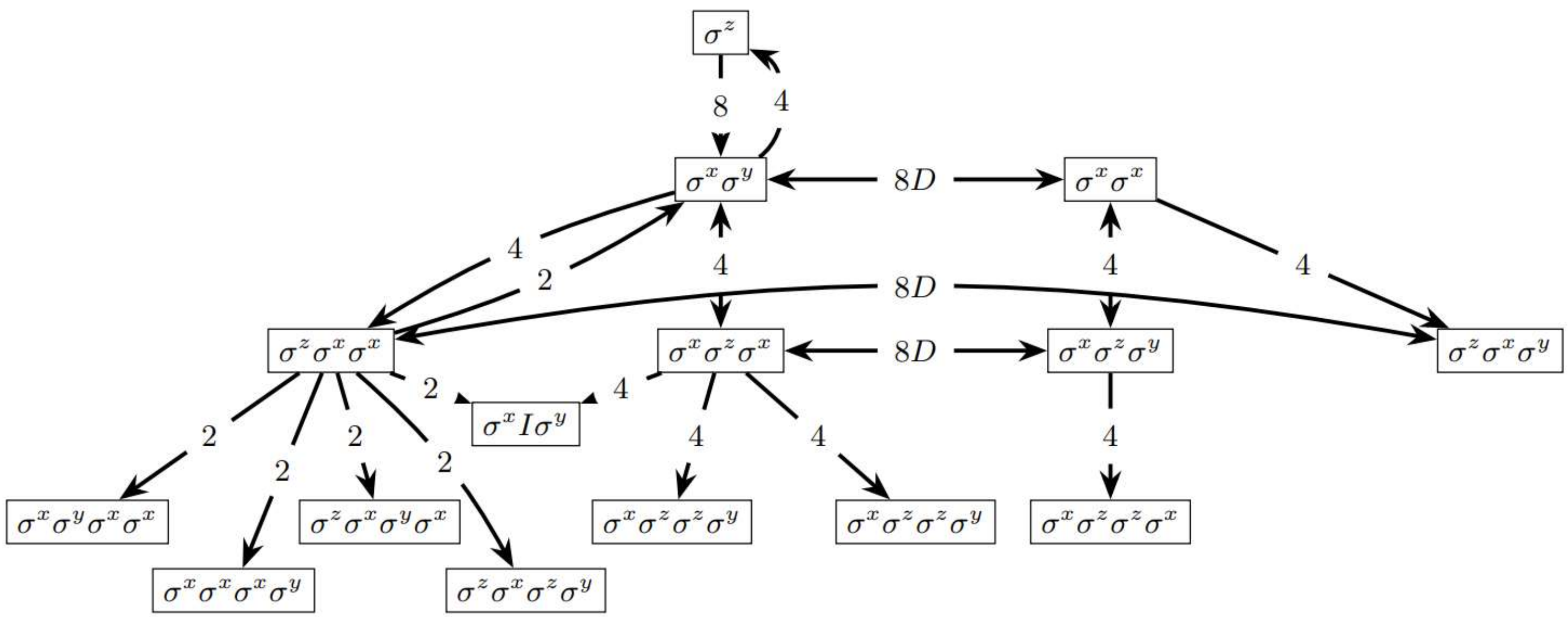


1-norm D=2
2-norm D=2
1-norm D=10
2-norm D=10
bound 8^k(D+1)^(k-1), D=2
bound 8^k(D+1)^(k-1), D=10
10^{200}
10^{150}
10^{100}
10^{50}
10^{0}
norm
0
25
50
75
100
k

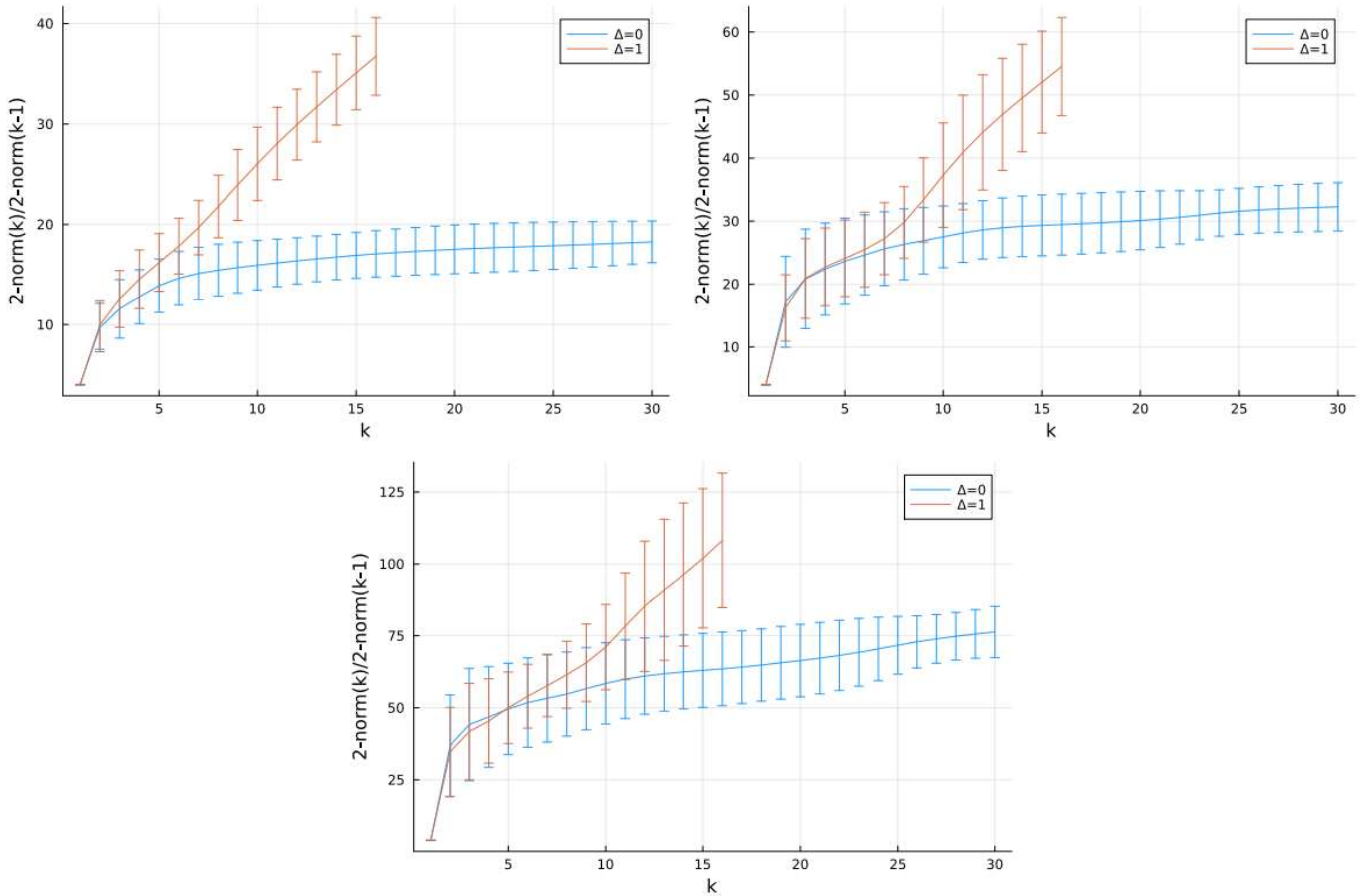

l=3
l=7
l=11
l=15
l=19
l=23
l=27
l=31
norm
10^{100}
10^{50}
10^{0}
0
25
50
75
100
k


l=3
l=7
l=11
l=15
l=19
l=23
l=27
l=31
norm
10^{200}
10^{150}
10^{100}
10^{50}
10^{0}
0
25
50
75
100
k

3. Graphical Language
c) Interacting case



- Exponential growth extremely unlikely…



40
30
20
10
2-norm(k)/2-norm(k-1)
5
10
15
20
25
30
k
Δ=0
Δ=1
60
50
40
30
20
10
2-norm(k)/2-norm(k-1)
5
10
15
20
25
30
k
Δ=0
Δ=1
125
100
75
50
25
2-norm(k)/2-norm(k-1)
5
10
15
20
25
30
k
Δ=0
Δ=1

Conclusions

- Operator growth is a promising method for studying localization
- Expect at most exponential growth in any localized phase, but our findings indicate that this is highly unlikely in the disordered Heisenberg model

References

- https://caneslocalisation.github.io/assets/resources/Papic_lecture1.pdf
- https://journals.aps.org/pr/pdf/10.1103/PhysRev.109.1492
- https://www.sciencedirect.com/science/article/pii/S0003491605002630
- https://arxiv.org/abs/1010.1992