Operator Growth and Many-Body Localization via Graphs and Nested Commutators

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Abstract

While many-body localization was first shown to exist by Anderson in 1958 for non-interacting, disordered lattice systems, an open question remains of whether such localization persists when electron interactions are included. If so, this would allow for exciting applications such as quantum memory and zero thermal conductivity. Most recent research in this area has been based on perturbative calculations and numerical results. This project aimed to use a new framework to describe localization in various quantum systems both numerically and analytically; in particular, operator growth was studied via nested commutators modelled by graph structures. These techniques allowed for a thorough study of the non-interacting Heisenberg model.

Quantum Heisenberg Model

- Start from model Hamiltonian: **Heisenberg model**

$$H = j\sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + h_i S_i^z\right)$$

- S_i^x, S_i^y, S_i^z = **spin-1/2 operators** acting on site i of one-dimensional lattice; h_i = **magnetic field (disorder)**

Models Studied	
***XX* Chain** (no disorder)	**Anderson Case** (*XX* + disorder)
• $\Delta = 0$ (non-interacting)	• $\Delta = 0$ (non-interacting)
• $h_i = 0\ \forall i$	• $h_i \in [-W, W]$
• Known to **thermalize**	• Known to **localize**

Thermalization, Localization, and Operator Growth

Thermalization	Localization
• Wave functions of electrons spread throughout crystal	• Wave functions exponentially localized near a particular site
• Local observables spread, become non-local	• Local observables retain memory of initial state

- **Baker-Campbell-Hausdorff formula**:

$$S_0^z(t) = e^{itH} S_0^z e^{-itH} = \sum_{k=0}^{\infty} [H, S_0^z]^k \frac{(it)^k}{k!}$$

$$[H, S_0^z]^k = \underbrace{[H, [H, \cdots [H, S_0^z]]]}_{k \text{ times}}$$

Norms to Measure Size of $[H, S_0^z]^k$	
1-norm	**2-norm**
$\lvert [H, S_0^z]^k \rvert_1$	$\lvert [H, S_0^z]^k \rvert_2$
• Sum of absolute values of pre-factors	• Sum of squares of absolute values of pre-factors

- **Goal**: analyze growth and structure of **nested commutator** $[H, S_0^z]^k$ with order k

Results: XX Case

- Commutator growth represented by the following **graph** structure (continues to infinity):

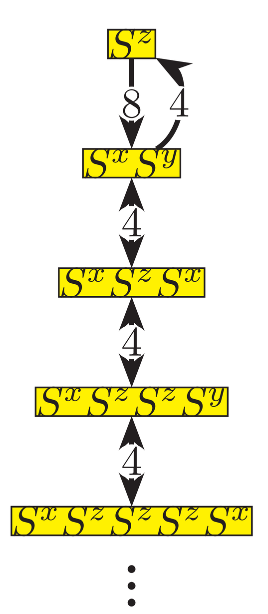

Figure 1: Graph for commutator growth, XX case.

Graph Representation	
Nodes/Vertices	• Terms in expanded commutator
Edges	• Connections between terms formed by commuting with Hamiltonian
Numbers on Edges	• 1-norm contribution of commuting result

- Terms which behave the same under commutation with Hamiltonian are grouped into a single node (e.g. terms are invariant under $S^x \Longleftrightarrow S^y$)

Terms

- Disjoint sites commute $\Rightarrow$ $(k+1)^2$ terms survive at each order

Norm Growth

$$|[H, S_0^z]^k|_1 = 8^k$$

$$|[H, S_0^z]^k|_2 = 2^{2k}\binom{2k}{k}^2 \approx \frac{2^{6k}}{\pi k}$$

Spatial Structure

- Support on l sites = contribution to norm of commutator from terms spanning l sites
- 1-norm support on one site:

$$s_1(k) = \begin{cases} 0 & \text{for odd } k \\ 2^{2k}\binom{k}{\frac{k}{2}} & \text{for even } k \end{cases}$$

$$\Rightarrow s_1(k) \approx \frac{2^{3k}}{\sqrt{\frac{k}{2}\pi}}$$

- 1-norm support on l sites: initially depends on l, but for $k \gg 1$, $s_l(k) \approx s_1(k) = \frac{2^{3k}}{\sqrt{\frac{k}{2}\pi}}$
- 2-norm support grows as $s_l(k)^2 = 2\frac{2^{6k}}{\pi k}$
- No decay of support $\Rightarrow$ operator spreading $\Rightarrow$ **thermalization**!

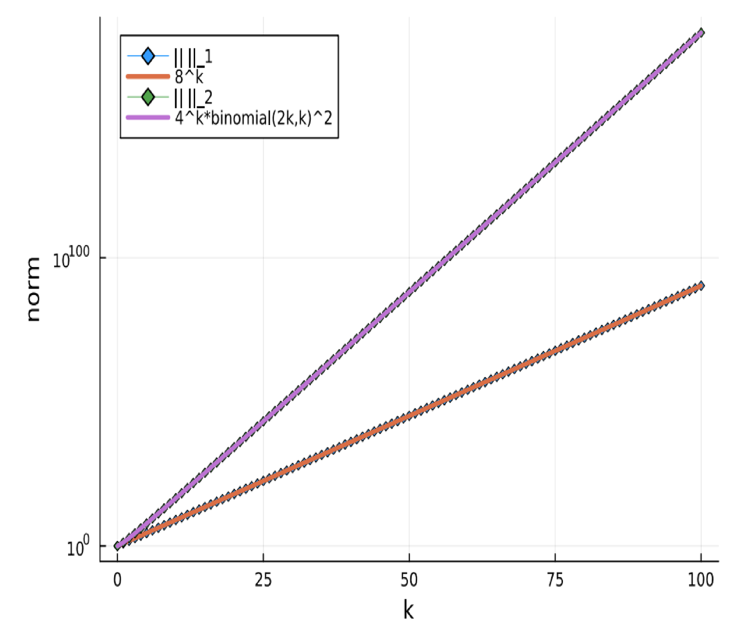

Figure 2: Numerical simulation shows that 1-norm and 2-norm of commutator matches analytical results of 8^k and $2^{2k}\binom{2k}{k}^2$ respectively. Both exhibit exponential growth asymptotically.

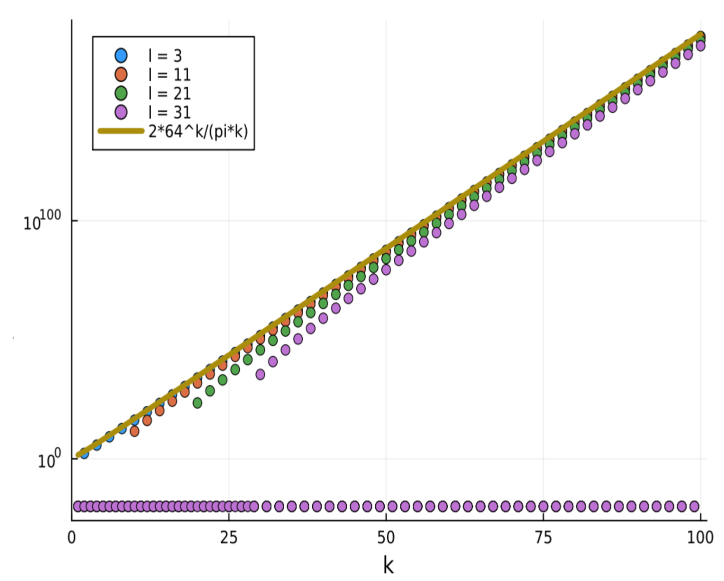

Figure 3: Numerical simulation shows that the 2-norm support grows as $2\frac{2^{6k}}{\pi k}$, verifying the analytical result. Since the asymptotic behaviour of the support on l sites is independent of l, the operator spreads throughout the system as time evolves: the system thermalizes.

Results: Anderson Case

- New $h_i S_i^z$ term in Hamiltonian $\Rightarrow$ new terms in the commutator $\Rightarrow$ new nodes and edges on the graph

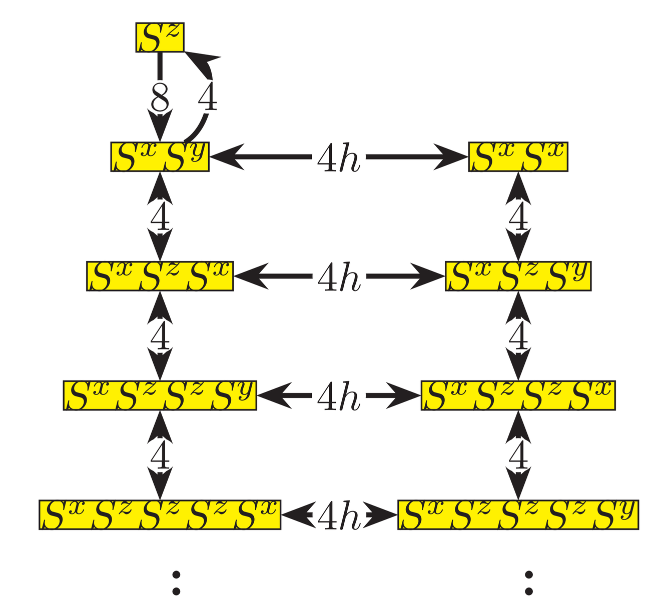

Figure 4: Graph for commutator growth, Anderson case.

- h = upper bound on magnetic field (disorder) strength

Definitions

- C_n = nth **Catalan number** = 1, 1, 2, 5, 14, 42, 132...

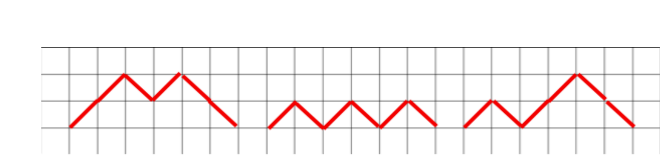

Figure 5: Example Dyck paths of length 6. C_n counts the number of Dyck paths of length $2n$. Gabriella Baracchini, MIT. (2016).

- ${}_2F_1$ = **hypergeometric function**

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

Norm Growth

$$|[H, S_0^z]^{(k)}|_1 \leq 2^{2k+1}(h+2)^{k-1}$$

Spatial Structure

- Tight upper bound on support on one site (even k):

$$s_1(k) \leq 2^{2k+1} \sum_{i=0}^{\frac{k-2}{2}} C_{\frac{k}{2}-i-1} \binom{k-1}{2i+1} h^{2i}$$

$$= 2^{2k+1} h^{k-2} {}_2F_1\left(\frac{1-k}{2}, \frac{2-k}{2}; 2; \frac{4}{h^2}\right)$$

- Looser bound on support on l sites:

$$s_l(k) \leq 2^{2k+1} \binom{k-1}{l-2} h^{k-l+1} \times {}_2F_1\left(\frac{1-k}{2}, \frac{2-k}{2}; 2; \frac{4}{h^2}\right)$$

- Ratio of support on l sites vs 1 site, in the asymptotic case of $k \gg l \gg 1$:

$$\frac{s_l(k)}{s_1(k)} \leq \binom{k-1}{l-2} h^{3-l} \approx \left(\frac{ke}{hl}\right)^l \frac{1}{\sqrt{2\pi l}} e^{\frac{-l^2}{2k}}$$

- Exponential decay of support compared to one site as l increases $\Rightarrow$ **localization**!
- As h increases, the decay gets faster

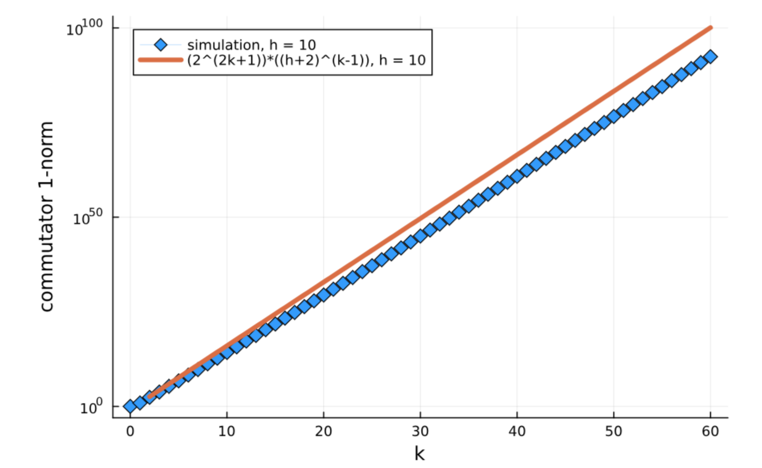

Figure 6: Verification of analytical bound on 1-norm via numerical simulation for the case of $h = 10$.

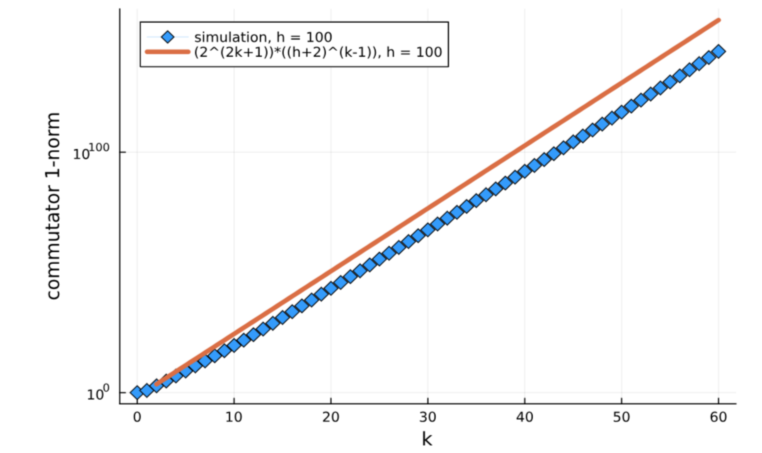

Figure 7: Verification of analytical bound on 1-norm via numerical simulation for the case of $h = 100$.

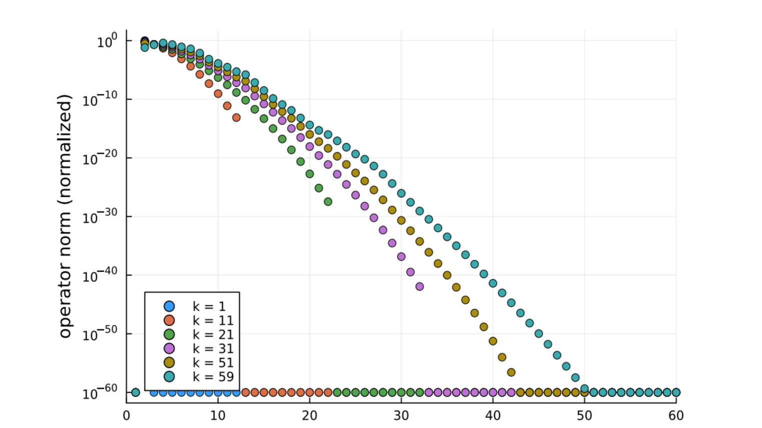

Figure 8: Numerical simulation in the case $h = 5$ shows that the support at distant sites clearly decays at least exponentially with constant k. This confirms the localization effect.

Conclusion and Future Work

By analyzing operator growth both numerically and analytically using graph modelling, we have been able to find strong theoretical evidence verifying thermalization in the non-disordered XX chain and localization in the Anderson case. Current efforts and progress aim to similarly analyze the interacting case, adding to the body of evidence supporting thermalization in the disordered Heisenberg model.

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