Many-Body Localization from an Operator Perspective

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<u>Reference</u>:
A. Weisse, R. Gerstner, &
J. Sirker, arXiv:2401.08031 (2024)

Operator Growth

Heisenberg (Euclidean) time evolution:

$$A(\tau) = e^{\tau H} A e^{-\tau H} = \sum_{k=0}^{\infty} [H,[H,[\dots,[H,A]]]] \frac{\tau^k}{k!}$$

$\Rightarrow$ Spreading of local operator A encoded in k-*fold commutator* $[H,A]^{(k)}$

$$||[H,A]^{(k)}|| \leq (2J)^k ||A|| B_k(2) \sim \left(\frac{k}{\ln k}\right)^k$$

General 1D nearest-neighbour model [Avdoshkin & Dymarsky, PRR (2020)]

Operator growth hypothesis: ergodic systems are characterized by maximal (*near-factorial)* norm growth [Parker et al., PRX (2019)]

Examples: Ising & Heisenberg

$$H = \sum_j (J_{i,i+1}\sigma_i^z\sigma_{i+1}^z + h_j^z\sigma_j^z + h_j^x\sigma_j^x)$$

$||[H, \sigma_0^z]^{(k)}|| \sim \left(\frac{k}{\ln k}\right)^k$ *proven* [Cao, JPA (2021)]

$$H = \sum_j (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z + h_j\sigma_j^z)$$

Symbolic calculations and graphical arguments indicate same scaling [arXiv: 2401.08031]

- *Maximal* growth ⇒ ergodic?
- No MBL? Or does operator growth hypothesis need to be modified?

Localization from an Operator Perspective

How can we define MBL using operator language?

1. A local operator A remains strictly exponentially localized to a finite region of space when commuted with the Hamiltonian H k times $[H, A]^{(k)}$.

MBL1

2. There exists a quasi-local unitary mapping from the microscopic Hamiltonian to an effective Hamiltonian written in terms of quasi-local charges.

$$\tilde{H} = UHU^{\dagger} = \sum_j \epsilon_j \tau_j^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \cdots$$

MBL2

Exponentially-decaying couplings

MBL1 Implies Exponential Total Norm Growth

Extending combinatorial method of [Avdoshkin & Dymarsky, PRR (2020)], and incorporating:

Hard cutoff for contributions of length $> \xi$:	*Exponential decay of contributions with increasing length:*
$\lvert\lvert [H, \sigma_0^z]^{(k)} \rvert\rvert \leq (2J)^k \lvert\lvert \sigma^z \rvert\rvert \frac{2^\xi \xi^k}{(\xi - 1)!}$	$\lvert\lvert [H, \sigma_0^z]^{(k)} \rvert\rvert \leq 2 \lvert\lvert \sigma^z \rvert\rvert (2\alpha J)^k$ $\alpha = -2e^{\frac{1}{\xi}} \ln\left(1 - e^{-\frac{1}{\xi}}\right)$

!!

Exponential total norm growth in localized model

➔

Immediate contradiction with (MBL?) Ising and Heisenberg?

- Let's examine further!

Total Norm Doesn't Give the Whole Story

Non-interacting XX and Anderson models:

$$H = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y\right)$$

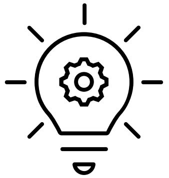
$$\Rightarrow ||[H, \sigma_0^z]^{(k)}|| = 8^k$$

$$H = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + 2h_j \sigma_j^z\right)$$

$$\Rightarrow ||[H, \sigma_0^z]^{(k)}|| \leq 8^k (D+1)^k$$
$$|h_j| \leq D$$

?

Both have exponential norm growth, but only one is localized



Look at local commutator norm $||[H, \sigma_0^z]_l^{(k)}|| = s_l(k)$.

Local Norm

$$H = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y\right)$$

$$\Rightarrow s_l^{(1)}(k) = 2^{2k+1} \binom{k}{\frac{k+1-l}{2}} \sim \sqrt{\frac{8}{k\,\pi}}\,8^k$$

Independent of l!

$$H = \sum_j \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + 2h_j \sigma_j^z\right)$$

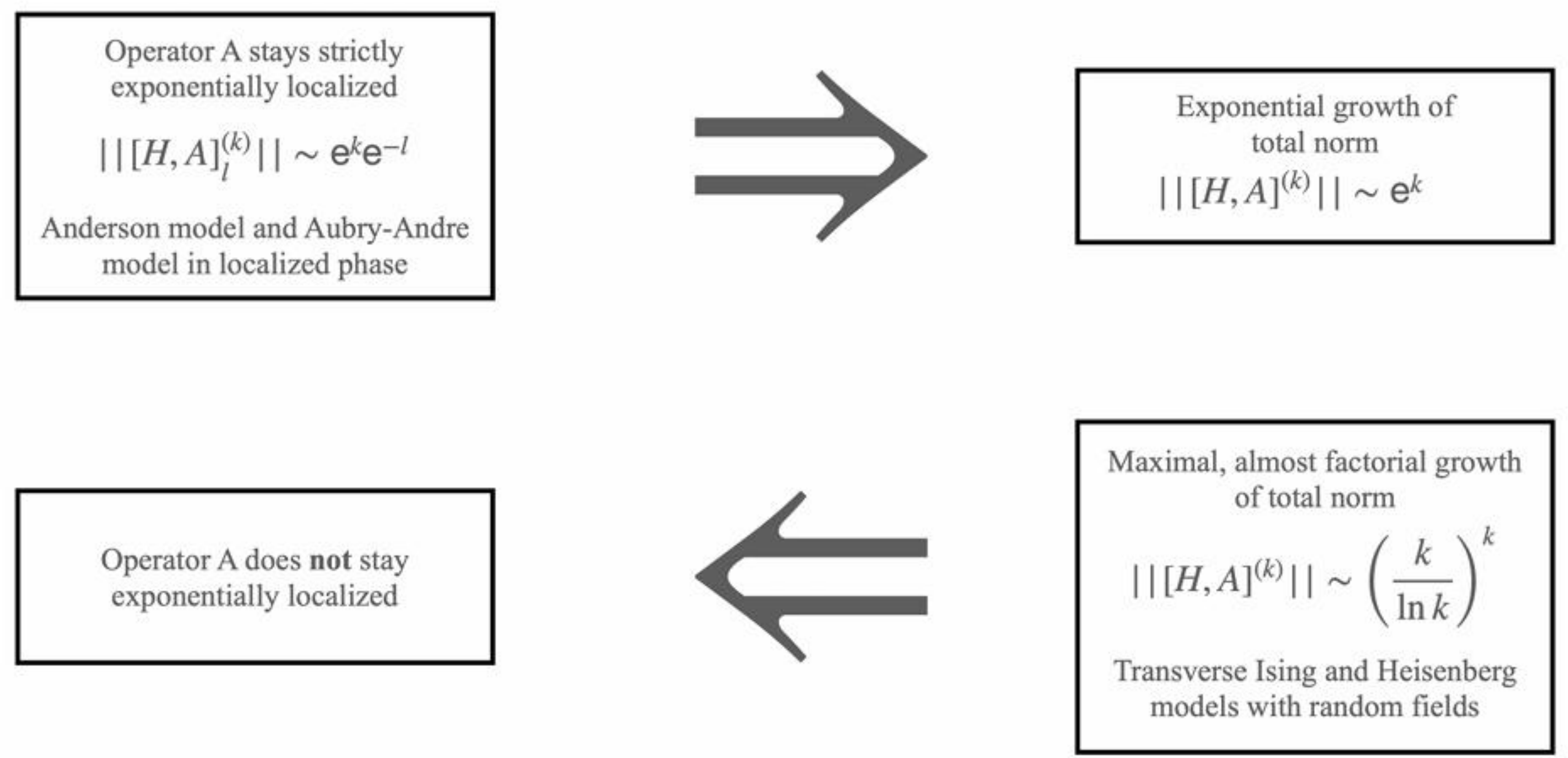
$$\Rightarrow s_l(k) \sim e^k D^{-l}$$

- Localization characterized by *exponential decay of local norm with increasing length*

!! Localized phase: $||[H,A]_l^{(k)}|| \sim e^k e^{-l}$

Full XXZ model lacks this decay with l [arXiv:2401.08031]

MBL1 is Incompatible with Ising/Heisenberg



(graphic created by J. Sirker)

MBL2: Effective Hamiltonian

$$\widetilde{H} = UHU^{\dagger} = \sum_j \epsilon_j \tau_j^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \cdots$$

- Since U is **unitary** and **quasi-local**:

1. *Local A mapped to* ***quasi-local*** $\tilde{A}$

2. *Invariant Frobenius norm:*

$$||[H, A]^{(k)}|| = ||[\widetilde{H}, \tilde{A}]^{(k)}||$$

3. $A = \sigma_0^z$

$\Rightarrow A^2 = \mathrm{I}$

$\Rightarrow \tilde{A}^2 = \mathrm{I}$

MBL2: Operators Delocalize?

- A can be mapped to a generic quasi-local operator:

$$\tilde{A} = \sum_{j,l,\{\alpha\}} e^{-\kappa_1|j|} e^{-\kappa_2 l} \sigma_j^{\alpha_0} \cdots \sigma_{j+l}^{\alpha_l}$$

$$||[\widetilde{H}, \tilde{A}]^{(k)}|| \leq \frac{4(2J)^k}{1-e^{-\kappa_1}} \sum_l (l+1)^k e^{-(\kappa_2 - \ln 4)l} \sim \frac{4e^{\tilde{\kappa}_2}}{\tilde{\kappa}_2(1-e^{-\kappa_1})} \left(\frac{2J}{\tilde{\kappa}_2}\right)^k k!$$

$$\Rightarrow s_l(k) \sim (Jl)^k e^{-\tilde{\kappa}_2 l}$$

Maximum at $l = \frac{k}{\tilde{\kappa}_2}$ → **Delocalizes** for large k

MBL2 cont'd

- Local operators can be mapped to *delocalizing* operators while satisfying all our known restrictions

Example:

$$\tilde{\sigma}^z = \sqrt{1 - \sum_l e^{-2\kappa}\, \sigma_0^z} + \sum_l e^{-\kappa l} \sigma_0^y \cdots \sigma_{l-1}^y \sigma_l^x$$

***Question**: Are there further (unknown) restrictions which lead to a stricter bound (possibly exponential?)*

If so, direct contradiction with Ising/Heisenberg due to norm invariance

U cannot exist !!

Conclusion

Two possible conclusions for Ising/Heisenberg:

$$\widetilde{H} = UHU^{\dagger} = \sum_{j} \epsilon_j \tau_j^z + \sum_{ij} J_{ij} \tau_i^z \tau_j^z + \cdots$$

1. $\widetilde{H}$ *exists*

- Unusual non-ergodic phase in which operators do not remain exponentially localized
- *Transport survives* [arXiv:2401.08031]
  ⇒ hard-pressed to call this MBL

2. $\widetilde{H}$ *does not exist*

- Supported by Schrieffer-Wolff transformation arguments [arXiv:2401.08031]